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# BRANCHING OF THE SOLUTIONS AND POLYNOMIAL INTEGRALS OF THE EQUATIONS OF DYNAMICS†

V. V. KOZLOV

Moscow

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The conditions for branching of the solutions of the equations of motion of natural mechanical systems in the complex time plane are obtained. The relation between the structure of the branching and the number of independent momentum-polynomial first integrals is investigated. Results of a general form are illustrated by examples from dynamics. 

The problem of the relation between the branching of the solutions in the complex time plane and the presence of nontrivial laws of conservation (often called first integrals or simply integral) goes back to the time of Painlevé. Golubev [1] related this problem to the classical investigations by Kovalevskaya, Lyapunov and Husson in rigid-body dynamics with a fixed point. The role of the branching solutions as an obstacle to integrability in the complex phase plane was investigated for the first time in [2], using the small Poincaré parameter method. These problems were then related to the self-intersection of the complex separatrices [3] and with the construction of a group of monodromic equations in variations [4]. Another approach is based on Lyapunov's method, developed for application to general quasi-homogeneous systems [5].

Below we establish a relation between the structure of the branching of the solutions as a function of the complex time and the number of momentum-polynomial integrals which enable the equations of dynamics to be employed.

## 1. MAIN RESULTS

Suppose $M^n$ is the configuration space of a dynamic system, and $P^{2n} = T^*M$ is its $2n$-dimensional phase space. We will denote the local coordinates in $M$ by $(x_1, \ldots, x_n) = x$; let $(y_1, \ldots, y_n) = y$ be conjugate momenta (Cartesian coordinates in the linear spaces $T_x^*M$). The variables $(x, y) = z$ are coordinates in phase space $P$.

Suppose

$$K(x,\ y) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(x) y_i y_j$$

is the kinetic energy of the system considered (a positive definite quadratic form in $T_x^*M$) and $F = (F_1(x), \ldots, F_n(x))$ is the force field (the covector field in $M$). If the forces $F$ are potential forces, we have

$$F_i = -\partial\Pi / \partial x_i, \quad 1 \leqslant i \leqslant n$$

where $\Pi$: $M^n \to \mathbb{R}$ is the potential energy.

The equations of motion have the following canonical form

$$\dot{x}_i = \partial K / \partial y_i, \quad \dot{y}_i = -\partial K / \partial x_i + F_i; \quad 1 \leqslant i \leqslant n \tag{1.1}$$

In the case of potential forces they take the form of Hamilton differential equations with Hamiltonian $H = K + \Pi$, which, of course, will be their integral. Equations (1.1) are invertible: they convert into one another with the involution $t \to -t, y \to -y$.

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All the known integrals of Eqs (1.1) are momentum polynomials (or functions of polynomials). The problem of the presence of polynomial integrals goes back to the time of Whittaker and Birkhoff [7]. Integrals that are linear in the momenta are related to hidden cyclic coordinates: after an appropriate replacement, one of the coordinates (say, $x_1$) does not occur the kinetic energy $K$ and the corresponding component of the force $F_1$ is equal to zero; then the initial linear integral will be identical with the momentum $y_1$. The existence of integrals that are quadratic in the momenta is related to the possibility of separating the variables. The problem of polynomial integrals of powers greater than or equal to three, is considerably more complex. A review of the results in this field can be found in [8, Chapter VIII].

In this paper the Whittaker–Birkhoff problem is considered from a simpler complex point of view. It is assumed that the manifold $M$ is provided with a complex structure, with respect to which $K$ and $F_1, \ldots, F_n$ are complex analytic functions in $P$ and $M$, respectively. We will investigate the problem of the presence of integrals that are polynomial in $y_1, \ldots, y_n$ with coefficients that are complex-analytic in $M$. Such integrals are often called single-valued polynomial integrals.

We will first put $F = 0$. We will then have the problem of the geodesics of the Riemann metric $K$ in $M$. It is well known (see, for example, [8]), that in this problem all the integrals can be assumed to be homogeneous polynomials in $y$: each homogeneous form of the expansion of the integral in a Maclaurin series in $y_1, \ldots, y_n$ will be an integral of the equations of the geodesics. The maximum number $s$ of such independent homogeneous integrals will be called the degree of Birkhoff integrability of the problem of geodesics. The degree of integrability is closely related to the topology of the configuration space. Suppose, for example, that $M^2$ is a compact oriented connected manifold. If the genus of $M$ is greater than unity, then $s = 1$ (the energy $H = K$ is a unique non-trivial integral), and for a torus (a surface of the first genus) $s \leqslant 2$. On the other hand, the geodesic flow on a standard two-dimensional sphere has three independent integrals (hence, here $s = 3$).

Suppose $\Phi : P \to \mathbb{R}$ is an analytic integral of the problem of geodesics. The derivative of $\Phi$, by virtue of the canonical equations (1.1), is

$$\dot{\Phi} = \sum \frac{\partial \Phi}{\partial y_i} F_i, \tag{1.2}$$

In the case of a potential force field

$$\dot{\Phi} = \{\Phi, \ \Pi\},$$

where $\{\cdot, \cdot\}$ is the standard Poisson bracket.

Suppose $t \to z_0(t)$ is one of the solutions of the shortened system (1.1), when $F_i = 0$. In view of the assumption of analyticity, $z_0(\cdot)$ is a holomorphic function of the complex time $t$, that is single-valued on a certain Riemann surface $\Omega$ ($\Omega$ is obtained as a result of the maximum analytic extension of its analytic element, the existence of which guarantees Cauchy's theorem). The composition $t \to \dot{\Phi}(z_0(t))$ is a holomorphic function on $\Omega$ or, in a certain subregion of it, if the components of the force $F$ have singularities. Suppose $\gamma$ is a closed oriented curve on $\Omega$, in the neighbourhood of which the function $t \to \dot{\Phi}(t)$ is holomorphic.

*Theorem* 1. If

$$\int_\gamma \dot{\Phi}(z_0(t))dt \neq 0, \tag{1.3}$$

the complete system of equations (1.1) has solutions that are multivalued on $\Omega$.

The branching property of the solutions means the following. By Cauchy's theorem, Eqs (1.1) have solutions which are holomorphic in the neighbourhood of the point $t_0 \in \gamma$, and when $t = t_0$ they take a specified value from $P$. Theorem 1 asserts that among these solutions there are those analytic attenuation of which along the closed path $\gamma$ leads to multivalued functions.

Theorem 1 is convenient to use in practice when the general solution of the problem of geodesics is represented by meromorphic functions in the complex time plane. Then the Riemann surface $\Omega$ is the complex plane $\mathbb{C} = \{t\}$, from which the poles of the meromorphic vector function $t \to z_0(t)$ are removed. Suppose the components of the force $F$ have no singularities. Then $f(t) = \dot{\Phi}(z_0(t))$ will be a meromorphic function. Theorem 1 asserts that if $f$ has at least one pole with non-zero residue, the solutions of system (1.1) will branch as a function of complex time.

One must, of course, bear in mind that in the general case $\dot{\Phi}(z(t)) \neq [\Phi(z(t))]$, otherwise integral (1.3) will be equal to zero by the Newton–Leibnitz formula.

The branching of the solutions is assumed to be related to the chaotic dynamics and the absence of integrals—conservation laws (see, for example, [9, 10]). However, not every branching is dangerous from the point of view of the property of integrability.

The simplest example is a one-dimensional Hamiltonian system with Hamiltonian $H = y^2/2 + h(x)$ where $h(x)$ is a polynomial of degree greater than or equal to five with simple roots. Almost all solutions are multivalued as functions of complex time, but the system allows of a polynomial integral $H$.

Suppose $s$ is the degree of Birkhoff integrability of the problem of geodesics of a Riemann manifold $(M, K)$ and $\Phi_1, \ldots, \Phi_s$ is a set of independent integrals, homogeneous with respect to the momenta. More exactly, it is assumed that the gradients of these functions are linearly independent at least at one point of the phase trajectory $t \to z_0(t)$. Then, it turns out that they are independent at all points of this trajectory (see, for example, [4]). The mapping $\Phi : P \to \mathbb{C}^s$, specified by the formula $z \to (\Phi_1(z), \ldots, \Phi_s(z))$, is called a momentum mapping.

We will calculate the derivative $\dot{\Phi}(z)$ by virtue of the complete system (1.1) (from formula (1.2)) and then form the composition $t \to \dot{\Phi}(z_0(t))$. As a result we obtain the vector function $\dot{\Phi}$, which is holomorphic on the Riemann surface $\Omega$.

Consider the natural homeomorphism

$$H_1(\Omega, \mathbb{Z}) \xrightarrow{\pi} \mathbb{C}^s$$

specified by the formula

$$\gamma \to \int_\gamma \dot{\Phi}(z_0(t))dt \tag{1.4}$$

Here $\gamma$ are one-dimensional cycles on $\Omega$; in view of the fact that $\dot{\Phi}$ is holomorphic the integrals (1.4) over homologous cycles are identical. Suppose $r$ is the rank of the group $\pi(H_1)$ as a system of vectors in $\mathbb{C}^5$ (the maximum number of linearly independent vectors from $\pi(H_1)$ over $\mathbb{C}$). For example, if $\dot{\Phi}(\cdot)$ is a meromorphic vector function on $\mathbb{C}$, the rank $r$ is equal to the maximum number of its linearly independent residues (as vectors from $\mathbb{C}^5$).

*Theorem* 2. We will assume that the system of equations (1.1) has $k$ polynomial holomorphic integrals with independent leading homogeneous forms. Then

$$k + r \leqslant s \tag{1.5}$$

It is easy to understand that the leading homogeneous forms of the integrals of system (1.1) are integrals of the problem of geodesics (when $F = 0$). The assumption that they are independent can obviously be removed since, in all cases we know of, one can indicate $k$ other polynomial integrals, in which the leading forms are independent almost everywhere. However, this assertion can only be proved in special cases. For example, when $M$ is an $n$-dimensional torus, $K = \Sigma a_{ij} y_i y_j /2$ is a non-degenerate quadratic form with constant coefficients [11]. Using Poincaré's method (see [8]), the assumption that the leading forms are independent can also be removed when $n - 1$ polynomial integrals of Eqs (1.1) with independent leading homogeneous forms are known, and the problem consists of finding one other polynomial integral. This situation certainly arises for Hamiltonian systems with two degrees of freedom: the energy integral $H = K + \Pi$ acts as the known integral.

Theorem 2 can be formulated particularly simply for conservative systems with two degrees of freedom ($n = 2$), the degree of integrability of which $s$ is equal to two. Here the problem may arise of the existence of an additional polynomial integral, independent of the energy integral. Theorem 2 gives a simple condition for non-integrability in the complex sense

$$\int_\gamma \{\Phi, \Pi\}(z_0(t))dt \neq 0 \tag{1.6}$$

Here $\Phi$ is the homogeneous integral of the geodesic problem.

Condition (1.6) can usefully be compared with the well-known condition of real non-integrability [8, Chapter IV]: we must take as the solution of the "unperturbed" problem the doubly asymptotic homoclinic solution, and the integration in (1.6) is carried out over the whole time axis $\mathbb{R} = \{t\}$. If this improper integral is non-zero, the complete system does not allow of a real polynomial integral with analytic coefficients, independent of the energy integral.

## 2. PROOF OF THEOREMS 1 AND 2

We will introduce the small parameter $\varepsilon$ into Eq. (1.1) by means of the substitution

$$x \to x, \quad y \to y/\sqrt{\varepsilon}, \quad t \to \sqrt{\varepsilon}t$$

Equations (1.1) will have the same form, except that the force $F$ must be replaced by $\varepsilon F$. When $\varepsilon = 0$ we will have the problem of inertial motion. The polynomial integrals of the momenta $y$ become polynomials in the parameter $\varepsilon$

$$\Phi_j(z, \varepsilon) = \Psi_j(z) + \varepsilon G_j(z) + \ldots \tag{2.1}$$

It is clear that $\Psi_j$ will be leading homogeneous forms of the initial integrals [8, Chapter II]. These functions are independent by assumption.

By Poincaré's theorem, the solutions of system (1.1) can be expanded in series in powers of $\varepsilon$

$$z(t, \varepsilon) = z_0(t) + \varepsilon z_1(t) + \ldots \tag{2.2}$$

which converge for small values of $\varepsilon$ uniformly with respect to $t$ from the neighbourhood of the closed curve $\gamma$.

We will prove Theorem 1. Suppose $\Phi$ is the integral of the unperturbed system. By (1.2) we have

$$\dot{\Phi} = \varepsilon \sum \frac{\partial \Phi}{\partial y_i} F_i \tag{2.3}$$

If all the solutions of the perturbed system are single valued on $\Omega$, the value of the function $\Phi$ as a function of time does not change after going round the closed contour $\gamma$. However, by (2.2) and (2.3) the increment of this function is equal to $\varepsilon I + o(\varepsilon)$, where $I$ is the integral on the left-hand side of (1.4). Consequently, for small $\varepsilon \neq 0$ solution (2.2) branches after analytic continuation along the closed curve $\gamma$. This is what was required.

We will now prove Theorem 2. Suppose $\omega \subset P$ is the transform of the Riemann surface $\Omega$ for the mapping $t \to z_0(t)$. Since $\Phi_1, \ldots, \Phi_s$ comprise the maximum set of independent integrals of the unperturbed system, at each point

$$\frac{\partial \Psi_j}{\partial z} = c_{j,1} \frac{\partial \Phi_1}{\partial z} + \ldots + c_{j,s} \frac{\partial \Phi_s}{\partial z}, \quad 1 \leqslant j \leqslant k \tag{2.4}$$

Since $z_0(\cdot)$ is a solution, the coefficients $c_j$ are constant on $\omega$ [8, Chapter IV]. Consequently

$$\frac{\partial \Psi_j}{\partial y} = \sum_{i=1}^{s} c_{j,i} \frac{\partial \Phi_i}{\partial y}$$

and, in particular,

$$\int_\gamma \dot{\Psi}_j(z_0(t))dt = \sum_i c_{j,i} \int_\gamma \dot{\Phi}_i(z_0(t))dt \tag{2.5}$$

Since (2.1) is a single-valued first integral of system (1.1), all the integrals on the left in (2.5) are equal to zero. Hence, by our assumption we obtain $r$ linearly independent vectors

$$\xi = \int_\gamma \dot{\Phi}(z_0(t))dt$$

such that $C\xi = 0$, $C = \|c_{j,i}\|$. But then rank $C \leqslant s - r$. By (2.4) the number of $k$ linearly independent vectors $\partial\Psi_j/\partial z$ is equal to the rank of $C$ and, consequently, inequality (1.5) holds: $k \leqslant s - r$.

## 3. SOME APPLICATIONS

**3.1.** We will consider the classical problem of the rotation of a heavy rigid body with a fixed point. The equations of motion allow of a Noethaer integral: the projection of the kinetic momentum on to the vertical is conserved. Assuming this projection to be zero and factorizing the rotations around the vertical over the group, we reduce the problem to an inverse system with two degrees of freedom ($n = 2$), the configuration space of which is a two-dimensional sphere (a Poisson sphere). The reduction of the order has been discussed in detail (for example, [12, Chapter III]).

If there are no forces we have an integrable Euler top. We know that if not all the principal moments of inertia coincide, for this problem $s = 2$ (incidentally, for the initial Euler system with three degrees of freedom $s = 4$). The solutions of the Euler problem are expressed in terms of $\Theta$-functions of the time $t$ and, consequently, are meromorphic functions of time in the complex plane $\mathbb{C} = \{t\}$ (see [6, Section 68]). If among the moments of inertia there are equal ones, the meromorphic functions degenerate into integer holomorphic functions of complex time.

We will take as the homogeneous integral $\Phi$ of Euler's problem, independent of the kinetic energy, the square of the length of the kinetic momentum vector of the top, and we take as the solution $z_0(\cdot)$ a doubly asymptotic trajectory, which approaches without limit to constant rotations of the body around the average axis of inertia in opposite directions. Of course, such solutions only exist for a dynamically asymmetrical body. The doubly asymptotic solution can be expressed in terms of elementary (but not elliptic) functions of time, and hence the further calculations are simplified considerably.

We calculated [13] the value of the Poisson bracket

$$\{\Phi, \Pi\}(z_0(t)) \tag{3.1}$$

This meromorphic function always has poles with non-zero residues. So, by Theorem 2 the reduced equations of the rotation of a heavy dynamically asymmetrical top does not allow of a single-valued polynomial integral, independent of the energy integral. This result was obtained for the first time in [14] by another method and was developed by Ziglin [4, 13].

The doubly asymptotic solutions in the Euler problem are heteroclinic. Hence, to prove the real non-integrability of the insufficient condition that the integral of (3.1) along the axis $\mathbb{R} = \{t\}$ is non-zero, it is required that it should take different values on the family of doubly asymptotic trajectories [13]. This condition reduces to the fact that the sum of the poles of the meromorphic function (3.1) in a certain strip close to the real axis is non-zero. To prove the simpler fact of the complex non-integrability it is sufficient to have at least one non-zero residue.

This example is of some historic interest: the classical results obtained by Kovalevskaya and Lyapunov on the single-valued solutions of the equations of the rotation of a heavy top led to the formulation of the general problem of the relation between the branching of the solutions of the equations of dynamics and the presence of single-valued integrals—the laws of conservation (see [1, 8]).

**3.2.** Suppose now that $M$ is an $n$-dimensional torus, $\mathbb{T}^n = \{x_1, \ldots, x_n \bmod 2\pi\}$, the kinetic energy

$$T = \frac{1}{2}\sum g_{ij}\dot{x}_i\dot{x}_j \tag{3.2}$$

is a non-degenerate quadratic form with constant coefficients, and the components of the force $f$ are analytic on $\mathbb{T}^n$ and are extended to meromorphic functions in the affine space of complex variables. The problem of the single-valued polynomial integrals of this system was considered previously in [15].

In view of the non-degeneracy of the form (3.2), the degree of Birkhoff integrability of the problem of geodesics is equal to $n$. The set of $n$ independent polynomial integrals of the problem of geodesics are the momenta

$$y_1, \ldots, y_n; \quad y_j = \partial T/\partial \dot{x}_j = \sum g_{ji}\dot{x}_i$$

Suppose

$$x = at + b, \quad a, b \in \mathbb{C}^n, \quad t \in \mathbb{C} \tag{3.3}$$

is a straight line from $\mathbb{C}^n$, which specifies one of the inertial motions. We will assume that the limitations of the meromorphic functions $F_1, \ldots, F_n$ on the straight line (3.3) are mesomorphic functions in the complex-time plane $\mathbb{C} = \{t\}$. We will denote them by $(f_1, \ldots, f_n) = f$.

By Theorem 1, if for certain $a$ and $b$ the function $t \to f(t)$ has a strip with non-zero residue, the general solution of the system of equations

$$\dot{x}_i = \partial H / \partial y_i, \quad \dot{y}_i = -\partial H / \partial x_i + F_i, \quad 1 \leq i \leq n; \quad H = T|_{\dot{x} \to y} \tag{3.4}$$

is branched in the plane $\mathbb{C} = \{t\}$.

Suppose that for certain $a$, $b \in \mathbb{C}^n$, the function $f$ has $m$ poles, the residues in which are linearly independent on $\mathbb{C}$, while system (3.4) allows of $k$ single-valued independent momentum-polynomial integrals. Then, by Theorem 2, $m + k \leq n$.

These assertions are proved in [15]. Theorems 1 and 2 are an extension of the results of [15] to reversible analytic systems of general form. Analogues of Theorems 1 and 2 for systems with configuration space $S^n$ were obtained in [16] by the same method.

**3.3.** As an example, which shows the effectiveness of Theorem 2, we will prove the complex non-integrability of the problem of the sliding of a point over an inclined ellipsoid of revolution (its axis of symmetry is not vertical). This result is new.

Suppose

$$(x^2 + y^2)/a^2 + z^2/b^2 = 1 \tag{3.5}$$

is the equation of the ellipsoid of revolution and $\Pi = \alpha x + \beta z$ ($\alpha$, $\beta$ = const) is the potential energy of the gravitational force. If $\alpha = 0$, the problem will be integrable: in addition to the energy, the angular momentum of the particle about the vertical is conserved

$$\Phi = x\dot{y} - y\dot{x} \tag{3.6}$$

The equations of motion have the form

$$\ddot{x} = \lambda x / a^2 - \alpha, \quad \ddot{y} = \lambda y / a^2, \quad \ddot{z} = \lambda z / b^2 - \beta \tag{3.7}$$

Here $\lambda$ is a Lagrange multiplier; taking into account the connection equation (3.5) it can be represented in the form of an explicit function of $\dot{x}, \dot{y}, \dot{z}$ and $x, y, z$.

Suppose $a \neq b$ (otherwise we will have the well-known integrable problem of a spherical pendulum). Then $s = 2$ (if $a = b$, obviously $s = 3$). We will take as the homogeneous integral of the problem of inertial motion the integral of the moment (3.6). Taking Eqs (3.7) into account we obtain the formula $\dot{\Phi} = \alpha y$.

We will introduce the natural parametrization of the surface of the ellipsoid (3.5) by the angular variables

$$x = a \sin \vartheta \sin \varphi, \quad y = a \sin \vartheta \cos \varphi, \quad z = b \cos \vartheta \tag{3.8}$$

Consider the closed geodesic on (3.5), which corresponds to the meridional section $x = 0$ (or $\varphi = 0$). The variable $\vartheta$ satisfies the obvious equation

$$(a^2 \cos^2 \vartheta + b^2 \sin^2 \vartheta)\dot{\vartheta}^2 = h \tag{3.9}$$

where $h$ is twice the kinetic energy. It is convenient to introduce the new variable $u = \cos\vartheta$ and to change to a new time $\tau$ by the formula

$$dt = (a^2 \cos^2 \vartheta + b^2 \sin^2 \vartheta) d\tau \tag{3.10}$$

From (3.9) and (3.10) we obtain

$$\frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = d\zeta, \quad \zeta = b\sqrt{h}\tau, \quad k^2 = \frac{b^2-a^2}{b^2}$$

Consequently, $u$ is the elliptic function of the variable $\zeta$ with modulus $k \neq 0$: $u = \mathrm{sn}\,(\zeta, k)$. By (3.8) $y = a\,\mathrm{cn}\,\zeta$.

By Theorem 2, we must integrate the 1-form $y(t)dt$ over the non-homologous zero cycle on the Riemann surface of the geodesic considered. After making the replacement $t \to \tau$ we obtain the 1-form $y(t(\tau))t'd\tau$. Apart from a non-zero factor, it has the explicit form

$$\mathrm{cn}\,\zeta\,\mathrm{dn}^2\,\zeta d\zeta$$

At the point $\zeta = iK'$($K'$ is the complete elliptic integral with additional modulus $k' = (1-k^2)^{1/2}$) we have a pole. Using the formulae ([17, Chapter 22])

$$\mathrm{cn}(\zeta + iK') = -\frac{i}{k\zeta} + \frac{2k^2-1}{6k}i\zeta + O(\zeta^3)$$

$$\mathrm{dn}(\zeta + iK') = -\frac{i}{\zeta} + \frac{2-k^2}{6}i\zeta + O(\zeta^3)$$

we find the residue. It is equal to $-i/(2k)$ and, consequently, is non-zero. Hence, the problem of the motion of a heavy particle over an inclined ellipsoid does not, in fact, have an additional holomorphic integral in the form of a velocity polynomial.

## 4. IRREVERSIBLE SYSTEMS

The results of Section 1 can be transferred to the more general case when additional gyroscopic forces $\Gamma y$, that are linear in the momenta, act on the system. The components of the matrix $\Gamma = \|\,\gamma_{ij}\,\|$ are assumed to be holomorphic functions on the complex manifold $M^n$.

Suppose again that $\Phi$ is a set of $s$ independent integrals of the problem of the inertial motion. The derivative (1.2) must be replaced by

$$\dot{\Phi} = \sum_{i,j} \frac{\partial \Phi}{\partial y_i}\gamma_{ij}y_j \tag{4.1}$$

It can be shown that Theorems 1 and 2 remain true after (1.2) is replaced by (4.1). The proof uses the substitution

$$x \to x, \qquad y \to y/\varepsilon, \qquad t \to \varepsilon t$$

Equations (1.1) do not change their form but the forces acting on the system will have the form $\varepsilon\Gamma y + \varepsilon^2 F$. The momentum-polynomial integrals become polynomials in $\varepsilon$.

As an example we will consider the problem of the motion of a particle of unit mass over the ellipsoid (3.5), which rotates with constant angular velocity $\omega$ around the $y$ axis. Derivative (4.1) of integral (3.6) of the unperturbed problem has the form

$$\dot{\Phi} = -\omega y\dot{z}$$

As in Section 3.3, we will consider the closed geodesic corresponding to the meridional section $x = 0$. If $a \neq b$, then, apart from an unimportant constant factor, the 1-form $y\dot{z}dt$ has the explicit form

$$\mathrm{cn}^2\,\zeta\,\mathrm{dn}\,\zeta d\zeta$$

The integral of this form over a small circle surrounding the point $iK'$ is non-zero. Consequently, when $a \neq b$, the problem in question does not allow of a single-valued polynomial integral independent of the energy integral.

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